A CHANNEL ABOVE A FLOW OF GROUNDWATER ALONG AN INCLINED WATER TABLE

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ABSTRACT: Formulas due to Numerov [2] for steady-state infiltration into a homogeneous isotropic soil have been used to compute the effects of the physical parameters on the infiltration characteristics.

1. Formulation of the problem and form of solution. It sometimes happens that a sloping impermeable horizon lies under soil containing groundwater; the tilt causes the groundwater to flow, and the surface of the water table in an unbounded region in the steady state is parallel to the underlying horizon ([1], chapter X, §5). The form of the flow alters if there are irrigation channels or ditches; in the first case there is influx, and in the second a lowering of the water table near the ditch. Here this problem is considered mainly for a channel (irrigation).

If the water in the channel (drain) is shallow, Numerov's solution [2] via the theory of functions gives the dependence of the complex coordinate $z = x + iy$ and of the complex potential $\omega = \varphi + i\psi$ (in which φ is the velocity potential and ψ is the current function) on the complex parameter $\zeta = \xi + i\eta$, which varies within a half-plane. Here we have to distinguish the three possibilities illustrated in Figs. 1–3.

(a) The water enters the soil throughout the width of the channel. The left branch in the free surface has its minimum at E, while the streamline EC separates the groundwater from the water entering from the channel. This case arises if the parameters of the problem are as follows [1]:

$$h < h_1 \sqrt{1-\beta}, \tag{1.1}$$

in which h is the depth of the flow to the left at infinity (equal to the depth of the groundwater flow in the absence of the channel), which (following Numerov) we call natural flow, while h_1 is the depth of the flow to the right at infinity and β characterizes the position of point B on the real axis of plane ζ (Fig. 3).

(b) The water enters the soil through a part EB of the bottom of the channel (Fig. 1b) while part AE drains the groundwater flow, and along KA (the drainage area) the groundwaters pass downward and upward. The rest of the flow, which is bounded from above by the streamline DEC, passes by the channel down the impermeable horizon and mixes with the water entering from the channel. The condition for this to occur is

$$h_1\sqrt{1-\beta} < h < h_1/\sqrt{1-\beta}. \tag{1.2}$$

(c) Flow to the channel occurs from both sides in such a way that we may term it a drain (Fig. 1c). This case occurs when

$$h > h_1/\sqrt{1-\beta}, \tag{1.3}$$

the right branch BC of the water table has a maximum, and drainage areas exist on both sides of AB. The part of the flow bounded from above by DEC enters the drain.

Numerov's solution covers all three cases and takes the following form when the coordinate systems are chosen in accordance with Figs. 1–3:

$$z = -T \operatorname{ctg} \pi\alpha + q + i\omega -$$
$$-\frac{\cos\pi\alpha}{\pi}\zeta^{\alpha}(\zeta-1)^{1-\alpha}\int_0^1 \frac{t^{-\alpha}(1-t)^{\alpha-1}\varphi(t)}{t-\zeta}\,dt,$$
$$\omega = i(q-q_1) + \frac{2}{\pi} q_1 \operatorname{ar\,ch}\sqrt{\frac{1-\beta}{\beta(\zeta-1)}} - \frac{2}{\pi} q \operatorname{ar\,ch}\sqrt{\frac{1}{\beta\zeta}}, \tag{1.4}$$

$$\varphi(t) = \frac{2}{\pi} q_1 \operatorname{ar\,sh}\sqrt{\frac{1-\beta}{\beta(1-t)}} - \frac{2}{\pi} q \operatorname{ar\,ch}\sqrt{\frac{1}{\beta t}},$$
$$q = h\sin\pi\alpha\cos\pi\alpha, \qquad q_1 = h_1\sin\pi\alpha\cos\pi\alpha. \tag{1.5}$$

Here T is the depth of the impermeable horizon from the bottom of the channel under the origin, $\pi\alpha$ is the angle to the horizontal of that horizon, and q and q_1 are the reduced flow rates in the upper and lower parts, respectively, which are related to h and h_1 by (1.5).

We consider T, α, and h (or q) as initial parameters to be specified in the solution of (1.4), while h_1 and the quantity q_1 in the second

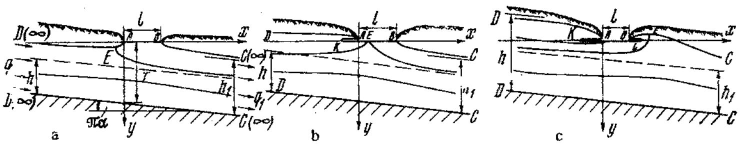

Fig. 1. Region of infiltration z.

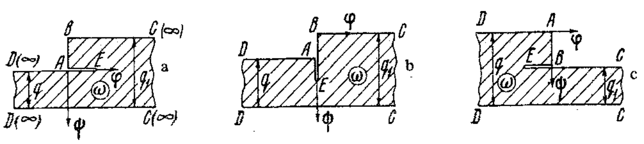

Fig. 2. Region of the complex potential ω.

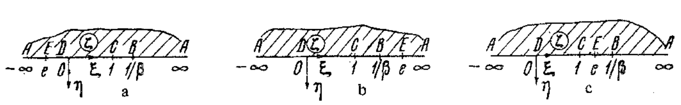

Fig. 3. Accessory half-plane of the complex variable ζ.

equation in (1.5) are to be determined, themselves serving to define the reduced infiltration flow rate q_0 per unit length of channel:

$$q_0 = q_1 - q = (h_1 - h) \sin \pi\alpha \cos \pi\alpha. \tag{1.6}$$

Also, parameter β of the conformal representation is unknown. Numerov used the following two conditions (Figs. 1 and 3) to determine this:

$$z(\infty) = 0, \quad z(1/\beta) = l. \tag{1.7}$$

From the second equation in (1.3) we have

$$\omega(\infty) = 0, \quad \omega(1/\beta) = i(q - q_1). \tag{1.8}$$

The first formula in (1.4) is used with (1.5), (1.7), and (1.8) to get

$$h - T\frac{1}{\sin^2\pi\alpha} + \frac{2\cos\pi\alpha}{\pi^2}\int_0^1 \frac{(1-t)^{\alpha-1}}{t^\alpha} f(t)\,dt = 0, \tag{1.9}$$

$$\sin\pi\alpha\cos\pi\alpha\left[h_1 - T\frac{1}{\sin^2\pi\alpha} + \right.$$

$$\left. + \frac{2\cos\pi\alpha}{\pi^2}(1-\beta)^{1-\alpha}\int_0^1 \frac{(1-t)^{\alpha-1} f(t)\,dt}{t^\alpha(1-\beta t)}\right] = l, \tag{1.10}$$

$$f(t) = h_1 \operatorname{ar\,sh}\sqrt{\frac{1-\beta}{\beta(1-t)}} - h \operatorname{ar\,ch}\sqrt{\frac{1}{\beta t}}. \tag{1.11}$$

2. **Calculations and preliminary transformations.** We solve (1.9) for h_1:

$$h_1 = \left[\frac{\pi_2}{\cos\pi\alpha}\left(\frac{T}{\sin^2\pi\alpha} - h\right) + \right.$$

$$\left. + 2h\int_0^1 \frac{(1-t)^{\alpha-1}}{t^\alpha} \operatorname{ar\,ch}\sqrt{\frac{1}{\beta t}}\,dt\right] \times$$

$$\times \left[2\int_0^1 \frac{(1-t)^{\alpha-1}}{t^\alpha} \operatorname{ar\,sh}\sqrt{\frac{1-\beta}{\beta(1-t)}}\right]^{-1}. \tag{2.1}$$

We substitute (2.1) into (1.10) to get an equation for β. We denote the left-hand side of this equation by $F(\beta)$ to get

$$F(\beta) = l. \tag{2.2}$$

The first step in the program we wrote for the M-20 computer is to determine from the interval (0.1) as the root of (2.2). This procedure is performed by division into halves ([3], chapter IV, §3), which is applicable if $F(\beta)$ is a monotonic function.

As $F(0) = 0$, $F(1) = \infty$, $F(\beta)$ increases as β goes from zero to one. We cannot determine whether β increases monotonically (which is equivalent to establishing the uniqueness of the solution to (2.2)) by direct differentiation of $F(\beta)$ with respect to β; however, we hypothesize that the increase in $F(\beta)$ is strictly monotonic on an intuitive basis and from calculation results, i.e., we assume that

$$\frac{dF(\beta)}{d\beta} = \frac{dl}{d\beta} > 0, \tag{2.3}$$

and we examine from the physical viewpoint some relationships derived from the above formulas.

We differentiate (1.9) and (1.11) with respect to β:

$$\int_0^1 \frac{(1-t)^{\alpha-1}}{t^\alpha}\frac{\partial f(\beta, t)}{\partial\beta}\,dt = 0 \tag{2.4}$$

$$\frac{\partial f(\beta, t)}{\partial\beta} = \frac{dh_1}{d\beta}\operatorname{ar\,sh}\sqrt{\frac{1-\beta}{\beta(1-t)}} +$$

$$+ \frac{1}{2\beta\sqrt{1-\beta t}}\left(h - \frac{h_1}{\sqrt{1-\beta}}\right). \tag{2.5}$$

It follows from (2.4) that $\partial f(\beta, t)/\partial\beta$ changes sign in the interval (0.1) while (2.5) indicates that this is possible only when $dh_1/d\beta$ and $h - h_1/(1-\beta)^{1/2}$ are opposite in sign, the other terms on the right in (2.5) being positive throughout the region $\{0 < t < 1) \times (0 < \beta < 1\}$. We also assume that (2.3) is correct to get

$$\frac{dh_1}{dl} > 0 \quad \text{for} \quad h < \frac{h_1}{\sqrt{1-\beta}},$$

$$\frac{dh_1}{dl} < 0 \quad \text{for} \quad h > \frac{h_1}{\sqrt{1-\beta}}. \tag{2.6}$$

Returning now to (1.1)-(1.3), we see that in cases (a) and (b) (filtration from the channel into the ground), h increases with l, i.e., as the infiltrating part of the channel expands. The second pair of inequalities in (2.5) corresponds to case (c) and is interpreted analogously: in bilateral flow to a drain, widening of the latter leads to greater uptake of groundwater, which is seen as reduced h_1. However, the fall in h_1 ceases at a certain l; this follows from the fact that $\beta \to 1$ for $l \to \infty$, and (2.1) gives $h_1 \to \infty$, and on further rise case (c) becomes case (b).

In the derivation of β we determine also h_1, and also q_0 from (1.6); then the program has provision for calculating the coordinates of the free surface. We omit the parametric equations for the free surface, which for the right branch BC ($1 < \zeta < 1/\beta$) are derived directly by separating the real and imaginary parts in the first equation of (1.4), and which for the left branch AD ($-\infty < \zeta < 0$) are obtained by first transforming the right-hand side of this equation while avoiding the singularities. One of the basic points is as follows in reducing the initial formulas to a form suitable for computer use.

Parameter ζ varies outside the range [0,1] for points on the free surface, and so the integral on the right in the first equation of (1.4) is not singular, but the integrand has singularities at the ends of the range of integration. The integral is calculated numerically (by Simpson's rule) on the computer, so these singularities must be avoided by isolating end parts $[0, \varepsilon]$ and $[1-\varepsilon, 1]$. The remaining interval $[\varepsilon, 1 - \varepsilon]$ is covered by the computation, while the integrals for $[0, \varepsilon]$ and $[1-\varepsilon, 1]$ are represented by approximate expressions.

Numerov's solution and the program based on it are suitable for all three cases of section 1. Special attention was given to cases characteristic of irrigation, where $h < T$, i.e., case (a) or (b) of section 1. For each particular case we specified $l^0 = l/T$, $h^0 = h/T$ and α, and thereby derived dimensionless values for the geometrical characteristics of the flow (assigned to T) and of the influx from the channel.

$$q_0{}^0 = q_0/T = Q_0/kT,$$

in which Q_0 is the flow from unit length of channel (m^3/day per meter) and k is the soil infiltration factor.

3. **Results.** The purpose was to examine q_0 and the shape of the free surface in relation to l^0, h^0, and α. The results are as follows.

1) Change in l^0 for a given T means change in width l, and this has very little effect on the infiltration characteristics under certain conditions, as Fig. 4 shows for $\log\eta$ as a function of $\log l^0$ for $h^0 = 0.2$ and α of 10^{-2}, 10^{-3}, 10^{-4}, and 10^{-5} via the data of Table 1.

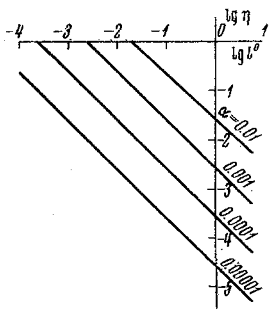

Fig. 4. Dependence of $\log\eta$ on $\log l^0$ for $h^0 = 0.2$.

Table 1

Relation between l^0, η and h_1^0 ($h^0 = 0.2$)

α	β	$6\cdot10^{-8}$	$24\cdot10^{-4}$	0.0156	0.125	0.354
10^{-5}	l^0	$2.513\cdot10^{-5}$	$1.036\cdot10^{-4}$	0.005037	0.04252	0.1389
	η	1.0000	0.2426	0.004989	$5.910\cdot10^{-4}$	$1.810\cdot10^{-4}$
	h_1^0	0.999856	0.999926	0.999956	0.999973	0.999985
10^{-4}	l^0	$2.509\cdot10^{-4}$	$3.287\cdot10^{-4}$	9.005261	0.04274	0.1391
	η	1.0000	0.7638	0.04774	0.005878	0.001807
	h_1^0	0.998567	0.999225	0.999558	0.999735	0.999846
10^{-3}	l^0	$2.469\cdot10^{-3}$	$2.566\cdot10^{-3}$	$7.486\cdot10^{-2}$	$4.486\cdot10^{-2}$	0.1410
	η	0.9999	0.9700	0.3339	$5.584\cdot10^{-2}$	$1.778\cdot10^{-2}$
	h_1^0	0.985845	0.992319	0.995610	0.997364	0.998464
10^{-2}	l^0	0.02130	0,02299	0.02858	$6.540\cdot10^{-2}$	0.1603
	η	0.9999	0.9969	0.8334	0.3720	0.1539
	h_1^0	0.878444	0.929881	0.958765	0.975009	0.985513
α	β	0.646	0.875	0.9844	0.99975586	0.999999941
10^{-5}	l^0	0.3310	0.6619	1.3239	2.6477	5.2957
	η	$7.594\cdot10^{-5}$	$3.797\cdot10^{-5}$	$1.899\cdot10^{-5}$	$9.496\cdot10^{-6}$	$4.747\cdot10^{-6}$
	h_1^0	0.999995	1.000007	1.000028	1.000069	1.000152
10^{-4}	l^0	0.3312	0.6621	1.3242	2.6489	5.2998
	η	$7.589\cdot10^{-4}$	$3.796\cdot10^{-4}$	$1.899\cdot10^{-4}$	$9.496\cdot10^{-5}$	$4.571\cdot10^{-5}$
	h_1^0	0.999949	1.000071	1.000281	1.000694	1.001526
10^{-3}	l^0	0.3330	0.6642	1.3280	2.6599	5,3408
	η	$7.542\cdot10^{-3}$	$3.787\cdot10^{-3}$	$1.899\cdot10^{-3}$	$9.531\cdot10^{-4}$	$4.796\cdot10^{-4}$
	h_1^0	0.999498	1.000713	1.002816	1.006966	1.015370
10^{-2}	l^0	0,3513	0.6849	1.3654	2.7734	5.7741
	η	$7.109\cdot10^{-2}$	$3.702\cdot10^{-2}$	$1.906\cdot10^{-2}$	$9.874\cdot10^{-3}$	$5.248\cdot10^{-3}$
	h_1^0	0.995587	1.007645	1.0289	1.072229	1.165203

Table 2

Relation between l^0, h_1^0, η and the Quantities x_δ (δ = 0.1, 0.01, 0.001)

	α	10^{-5}				10^{-4}			
	h^0	0.05	0.2	0.5	0.9	0.05	0,2	0.5	0,9
$\beta=6\cdot10^{-8}$	l^0	$2.984\cdot10^{-5}$	$2.513\cdot10^{-5}$	$1.571\cdot10^{-5}$	$3.142\cdot10^{-6}$	$2.979\cdot10^{-4}$	$2.509\cdot10^{-4}$	$1.568\cdot10^{-4}$	$3.136\cdot10^{-5}$
	h_1^0	0.9998	0.9999	0.9999	1.0000	0.9983	0.9986	0.9991	0.9998
	η	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1,0000
	$\|x_{0.1}\|$	30672	35544	38597	—	3062,3	3547.8	3843.1	—
	$\|x_{0.01}\|$	37205	53152	78297	69624	3705.9	5301.6	7804.1	6877.8
	$\|x_{0.001}\|$	41144	68069	115164	135761	4109.7	6810.6	11526	13665
$\beta=0.999999941$	l^0	5.2957	5.2957	5.2957	5.2957	6.2998	5.2998	5.2997	5.2997
	h_1^0	1,0002	1.0002	1.0002	1,0002	1.0015	1.0015	1,0015	1.0015
	η	$5.637\cdot10^{-6}$	$4.797\cdot10^{-6}$	$1.967\cdot10^{-6}$	$5.941\cdot10^{-7}$	$5.640\cdot10^{-5}$	$4.751\cdot10^{-5}$	$2.973\cdot10^{-5}$	$6.018\cdot10^{-6}$
	$\|x_{0.1}\|$	30677	35548	38599	—	3065.9	3572.9	3845.6	—
	$\|x_{0.01}\|$	37211	53157	78301	69625	3726.0	5331.3	7807.2	6942,6
	$\|x_{0.001}\|$	41150	68074	115167	135762	4114.8	6822.4	11529	13620

	α	10^{-3}				10^{-2}			
	h^0	0.05	0.2	0,5	0.9	0.05	0.2	0.5	0.2
$\beta=6\cdot10^{-8}$	l^0	0.002932	0.002469	0.001543	0.0003086	0.02529	0.02130	0.01331	0.002664
	h_1^0	0.9832	0.9858	0.9912	0.9082	0.8556	0.8784	0.9240	0.9848
	η	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	$\|x_{0.1}\|$	300.98	350.58	378.60	—	26,008	30.751	33,383	—
	$\|x_{0.01}\|$	366.23	525.81	773.21	683.11	32.334	48.508	73584	65.661
	$\|x_{0.001}\|$	405.78	674.63	1145.5	1350.9	36.234	63.714	112.30	135.12
$\beta=0.999999941$	l^0	5.3411	5.3408	5.3404	5.3397	5.7769	5.7741	5.7685	5.7810
	h_1^0	1.0154	1.0154	1.0154	1.0154	1.1653	1.1653	1.1650	1.1648
	η	$5.678\cdot10^{-4}$	$4.796\cdot10^{-4}$	$3.032\cdot10^{-4}$	$6.788\cdot10^{-5}$	0.00606	0.00525	0.00762	0.00144
	$\|x_{0.1}\|$	306.44	365.91	384.98	—	30.780	36.162	39.416	—
	$\|x_{0.01}\|$	371.81	779.56	695.98	37.183	37.183	53.353	78.418	74.449
	$\|x_{0.001}\|$	411.38	681.87	1151.81	1359.8	41.804	68.567	118,24	145.89

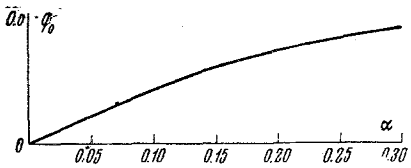

Fig. 5. Dependence of q_0^0 on α for l^0 = 0.5 and h^0 = 0.3.

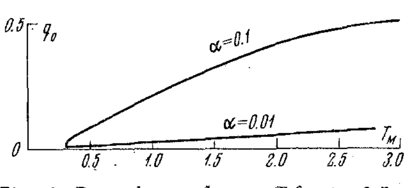

Fig. 6. Dependence of q_0 on T for l = 0.5 m and h = 0.3 m.

The quantity

$$\eta=\frac{q_0{}^0}{l^0}=\frac{Q_0}{kl}, \tag{3.1}$$

has been called [4] the flooding coefficient; it represents the ratio of the actual influx Q_0 to the flow rate kl in free infiltration.

The lines of Fig. 4 are almost straight, and the slopes differ little from −1; they are represented approximately by

$$\lg\eta\approx\lg A-(1-\mu)\lg l^0=\lg B-(1-\mu)\lg l, \tag{3.2}$$

in which B and μ are constants. The parameter B is discussed below, while μ is a quantity of the order of 0.0001–0.03 (increasing with α). The μ for the lines in Fig. 4 are 0.023, 0.002, 0.0002, and 0.0001.

We transform (3.2) to

$$\eta\approx Bl^{\mu-1}.$$

or, from (3.1),

$$q_0 \approx Bl^{\mu}. \tag{3.3}$$

Then (3.3) reflects directly the weak dependence of q_0 on l.

2) There is a linear dependence of q_0 on h, T, and α for $\alpha < 0.01$, and the following approximate relation applies:

$$q_0 \approx \pi\alpha\,(T - h). \tag{3.4}$$

Numerov [1] noted this, and our calculations confirm it completely. Relation (3.4) is illustrated by the dependence of q_0 on α (Fig. 5 for $l = 0.5$, $h = 0.3$, $T = 1$) and on T (Fig. 6 for $l = 0.5$ and $h = 0.3$). Figure 5 shows that the straight-line dependence of q_0 on α for small α gradually ceases to apply as α increases. In Fig. 6, $q_0(T)$ for $\alpha = 0.01$ is almost a straight line, while a curve applies for $\alpha = 0.1$. However, $\alpha = 0.1$ (inclination 18°) is outside the practical range of α.

Comparison of (1.6) and (3.4) shows that $h_1 \approx T$ for small α, which Numerov [1] noted and which is confirmed by Table 1. From (3.3) and (3.4) we get approximately that

$$q_0 \approx \pi\alpha\,(T - h)l^{\mu} \quad (B = \pi\alpha\,(T - h)), \tag{3.5}$$

in which μ is a small constant.

3) The left and right branches of the free surface have asymptotes parallel to the impermeable horizon and at distances h and h_1 from the latter (Fig. 1). The calculations showed that the free surface very nearly coincides with the asymptote on the right after a very short distance, whereas the depth exceeds the natural depth for a considerable distance on the left.

In the calculations on the left branch we determined the values x_δ of the x-coordinate for which $\Delta y < \delta$, in which Δy is the vertical distance between the free surface and the asymptote. Table 2 gives the absolute values (by virtue of the choice of coordinate system, $x_\delta < 0$) for δ of 0.1, 0.01, and 0.001 (the linear characteristics are referred to T).

The quantity Δy can serve as a measure of the water-table rise, while x_δ indicates the range of that rise. The latter concept is as arbitrary as that of the radius of action of a borehole when the depression funnel is of unlimited size. The dependence of x_δ on l is very slight, specially for small α, so the table gives only the largest and smallest values of $|x_\delta|$ for each of the values of α and h^0. The first corresponds to the minimum value of β out of those used in the calculations ($\beta \approx 6 \cdot 10^{-8}$), while the second represents the maximum value of β ($\approx 1-6 \cdot 10^{-8}$). Since $\delta = 0.1$ equals the depth of the natural flow for $h^0 = 0.9$, it cannot serve as a measure of the water-table rise, and so the corresponding columns of Table 2 do not contain $|x_{0.1}|$. The $|x_{0.1}|$ for the other h^0 in the table is of the order of $1/\pi\alpha$, which defines the distance from the channel to the point where the impermeable horizon meets the horizontal axis.

4) If $h = 0$, the calculation formulas are derived directly from the equations for the general case if we put $q = 0$ in these. The calculations confirm Numerov's approximate relations for the coordinates L and H of the point at which the free surface emerges on the impermeable layer in the upper part of the flow:

$$L \approx -T \operatorname{ctg} \pi\alpha, \quad H \approx 0.$$

These equations indicate that here the flooded zone extends to the point at which the impermeable layer meets the horizontal coordinate axis, and the free surface in that zone is nearly horizontal, with virtually no infiltration.

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